Algebraic foundations for inquisitive semantics

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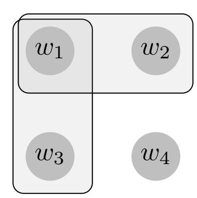
1 Introduction

Traditionally, meaning is identified with informative content. The central aim of inquisitive semantics [1, 2, 4, 5, a.o.] is to develop a notion of semantic meaning that embodies both informative and inquisitive content. To achieve this, the proposition expressed by a sentence φ, $[\varphi]$, is not taken to be a set of possible worlds, but rather a set of *possibilities*, where each possibility in turn is a set of possible worlds. In uttering a sentence φ, a speaker provides the information that the actual world is contained in at least one possibility in $[\varphi]$, and at the same time she requests enough information from other participants to establish for at least one possibility $\alpha \in [\varphi]$ that the actual world is contained in α.

Suppose, for instance, that $[\varphi] = \{\{w_1, w_2\}, \{w_1, w_3\}\}$, as depicted on the right. Then in uttering φ, a speaker provides the information that the actual world lies in $\bigcup[\varphi] = \{w_1, w_2, w_3\}$, and at the same time requests enough information to establish that the actual world lies in $\{w_1, w_2\}$ or to establish that it lies in $\{w_1, w_3\}$. In this way, $[\varphi]$ captures both the informative and the inquisitive content of φ.



As soon as the basic notion of meaning is enriched in this way, the question arises how the propositions expressed by complex sentences should be defined in terms of the propositions expressed by their constituents. In particular, if we limit ourselves to a first-order language, what is the role of connectives and quantifiers in this richer setting? That is, given two propositions that capture the informative and the inquisitive content of φ and ψ, how do we construct the propositions that suitably capture the informative and inquisitive content of $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\exists x.\varphi$, and $\forall x.\varphi$?

This issue has of course been addressed in previous work [1, 2, 4, 5, a.o.]. However, the clauses for connectives and quantifiers that have been formulated so far were motivated based on a limited range of linguistic examples. While

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such examples can be indicative of whether or not a semantics works the way it is supposed to work, they do not provide a proper foundation for the semantics.

The present paper develops an inquisitive semantics for a first-order language that is motivated by general algebraic concerns. As far as connectives are concerned it coincides with, and thus provides an algebraic foundation for the semantics that was developed in previous work. However, its treatment of quantifiers will diverge from previous work, and a careful assessment of these differences will deepen our understanding of the semantics.

The paper is structured as follows. In section 2 we briefly review the algebraic foundations of classical logic. In section 3 we develop an algebraically motivated inquisitive semantics, and in section 4 this semantics is compared with previous work. Section 5 concludes.

2 Algebraic foundations of classical logic

To illustrate our approach, let us briefly review the standard algebraic foundations of classical logic. In classical logic, the proposition expressed by a sentence φ, which we will denote as $[\varphi]$, is a set of possible worlds that captures the informative content of φ: in uttering φ, a speaker provides the information that the actual world is contained in $[\varphi]$. Given this classical way of thinking about propositions, there is a natural *order* between them: $A \leq_c B$ iff A provides at least as much information as B, i.e., iff $A \subseteq B$. This order in turn gives rise to certain algebraic operations on propositions. For instance, for any two propositions A and B, there is a proposition $M(A, B)$ that is the greatest lower bound of A and B relative to $\leq_c$, which, in algebraic jargon, is called the *meet* of A and B. Similarly, for every A and B, there is some proposition $J(A, B)$ that is the least upper bound of A and B relative to $\leq_c$, which is called the *join* of A and B. The existence of meets and joins implies that the set of all classical propositions Π_c, together with the order $\leq_c$, forms a *lattice*. This lattice is *bounded*: it has a bottom element, $\emptyset$, and a top element, W (the set of all possible worlds). Moreover, for every classical proposition A, there is another proposition $C(A)$ such that (i) the join of A and $C(A)$ coincides with the top element of the lattice, and (ii) the meet of A and $C(A)$ coincides with the bottom element of the lattice. $C(A)$ is called the *complement* of A, and the fact that every classical proposition has a complement implies that $\langle \Pi_c, \leq_c \rangle$ forms a *complemented lattice*. Finally, the meet and join operators *distribute* over each other, i.e., $M(A, J(B, B')) = J(M(A, B), M(A, B'))$ and $J(A, M(B, B')) = M(J(A, B), J(A, B'))$, which means that $\langle \Pi_c, \leq_c \rangle$ forms a distributive complemented lattice. Such lattices are also called *Boolean algebras*.

Now, the semantic operators we identified, *meet*, *join*, and *complementation*, can be associated with syntactic operators like *conjunction*, *disjunction*, and *negation*, respectively. That is, we can define:

- $[\neg\varphi] = C([\varphi])$
- $[\varphi \wedge \psi] = M([\varphi], [\psi])$
- $[\varphi \vee \psi] = J([\varphi], [\psi])$

Moreover, it can be shown that $C([\varphi])$ amounts to $W - [\varphi]$, that $M([\varphi], [\psi])$ amounts to $[\varphi] \cap [\psi]$, and that $J([\varphi], [\psi])$ amounts to $[\varphi] \cup [\psi]$. This yields the familiar clauses for negation, conjunction and disjunction:

- $[\neg\varphi] \quad = W - [\varphi]$
- $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$
- $[\varphi \vee \psi] = [\varphi] \cup [\psi]$

This is how classical propositional logic is obtained, and the approach can be extended to first-order logic [7, 8]. Notice that everything starts with a certain notion of propositions, and a natural order on these propositions. This order, then, gives rise to certain operations on propositions—*meet*, *join*, and *complementation*—and classical propositional logic is obtained by associating these semantic operations with the syntactic connectives $\wedge$, $\vee$, and $\neg$.

3 Algebraic inquisitive semantics

Exactly the same strategy can be applied in the inquisitive setting. Only now we have a richer notion of propositions, and a different *order* on them: propositions are not only ordered in terms of their informative content, but also in terms of their inquisitive content.

3.1 Ordering propositions

Let us first officially define what we take propositions to be in the inquisitive setting. In fact, the definition below is provisional—it will be slightly refined—but it forms the most natural point of departure.

Definition 1 (Possibilities and propositions).

- *A possibility is a set of possible worlds.*
- *A proposition is a non-empty set of possibilities.* [*to be refined*]

Propositions embody informative and inquisitive content in the following way. Someone who utters a sentence that expresses a proposition A provides the information that the actual world lies in at least one of the possibilities in A. Thus, she provides the information that the actual world lies in $\bigcup A$. We will refer to $\bigcup A$ as the *informative content* of A, and denote it as $\mathsf{info}(A)$.

Definition 2 (Informative content). $\mathit{info}(A) = \bigcup A$

On the other hand, someone who utters a sentence that expresses a proposition A also *requests* certain information from other conversational participants. Namely, she requests enough information to establish for at least one possibility $\alpha \in A$, that the actual world is contained in α.

Thus, propositions can be ordered in terms of their informative content, but also in terms of their inquisitive content. Just as in the classical setting, one

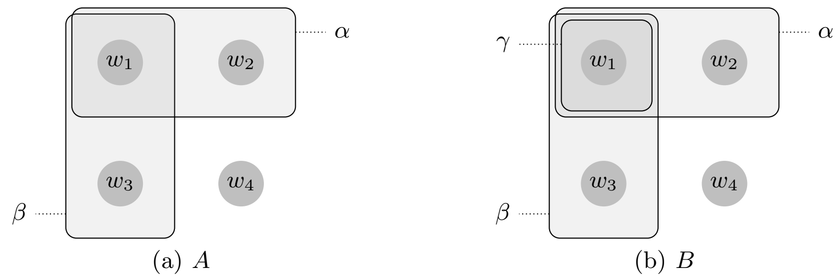

Fig. 1. Two non-identical propositions that are equivalent w.r.t. $\leq$.

proposition A is at least as informative as another proposition B, $A \leq_{\mathsf{info}} B$, just in case $\mathsf{info}(A) \subseteq \mathsf{info}(B)$. As for inquisitiveness, we say that one proposition is at least as inquisitive as another proposition B, $A \leq_{\mathsf{inq}} B$, iff A requests at least as much information as B, i.e., iff every response that satisfies the request issued by A also satisfies the request issued by B. This means that every response that provides enough information to establish some possibility in A must also provide enough information to establish some possibility in B. This holds just in case every possibility in A is contained in some possibility in B. Thus, $A \leq_{\mathsf{inq}} B$ if and only if $\forall\alpha \in A.\ \exists\beta \in B.\ \alpha \subseteq \beta$. These two orders can be combined into one overall order: $A \leq B$ iff both $A \leq_{\mathsf{info}} B$ and $A \leq_{\mathsf{inq}} B$. In sum:

Definition 3 (Ordering propositions).

– $A \leq_{\mathsf{info}} B$ *iff* $\mathsf{info}(A) \subseteq \mathsf{info}(B)$
– $A \leq_{\mathsf{inq}} B$ *iff* $\forall\alpha \in A.\ \exists\beta \in B.\ \alpha \subseteq \beta$
– $A \leq B$ *iff* $A \leq_{\mathsf{info}} B$ *and* $A \leq_{\mathsf{inq}} B$

Now, notice that $A \leq_{\mathsf{inq}} B$ actually *implies* that $A \leq_{\mathsf{info}} B$. After all, if every possibility in A is contained in some possibility in B, then $\bigcup A$ must also be contained in $\bigcup B$. This means that $A \leq B$ if and only if $A \leq_{\mathsf{inq}} B$.

Fact 1 (Simplified overall order) $A \leq B$ *iff* $\forall\alpha \in A.\ \exists\beta \in B.\ \alpha \subseteq \beta$

Now let us see whether $\leq$ forms a *partial order*, i.e., whether it is reflexive, transitive, and anti-symmetric. It is easy to see that $\leq$ is indeed reflexive and transitive. However, $\leq$ is *not* anti-symmetric. That is, it is possible to find two propositions A and B such that $A \leq B$ and $B \leq A$, but $A \neq B$. Two such propositions are depicted in figure 1: the proposition depicted on the left, A, consists of two possibilities, α and β, while the proposition depicted on the right consists of three possibilities, α, β, and γ. Thus, these two propositions are not identical. However, they are equivalent w.r.t. $\leq$. That is, they are just as informative and inquisitive.

To see this, first notice that $\mathsf{info}(A)$ and $\mathsf{info}(B)$, i.e., the union of the possibilities in A and the union of the possibilities in B, clearly coincide. Thus, A and B provide just as much information. To see that A and B also request just as much information, consider a response that satisfies the request issued by A. Such a response must either provide the information that the actual world lies in α or it must provide the information that the actual world lies in β. But that means that this response would also satisfy the request issued by B. And vice versa, any response that satisfies the request issued by B also satisfies the request issued by B. Thus, A and B request just as much information. In other words, A and B are just as inquisitive.

This shows that, as long as we are just interested in capturing informative and inquisitive content, we don't really need to define propositions as *arbitrary* sets of possibilities. Rather, we could adopt a more restricted notion of propositions, such that any two propositions that are non-identical also really differ in informative and/or inquisitive content.

To this end, we will define propositions as *persistent* sets of possibilities.

Definition 4 (Propositions as persistent sets of possibilities).

- *A set of possibilities A is persistent if and only if for every $\alpha \in A$ and every $\beta \subseteq \alpha$, we also have that $\beta \in A$.*
- *A proposition is a non-empty, persistent set of possibilities.*
- *The set of all propositions is denoted by Π.*

To see that persistency is a natural constraint on propositions in the present setting, consider the following. We are conceiving of propositions as sets of possibilities, and not just as sets of possible worlds, because in this way it is possible for a proposition to embody a certain issue. An issue can be characterized by the range of responses that resolve it. Thus far, we have been assuming the following relationship between the responses that resolve the issue embodied by a proposition A and the possibilities that A consists of: a response resolves the issue embodied by A iff its informative content is contained in some possibility $\alpha \in A$. But we could also think of this relationship slightly differently, namely we could say that a response resolves the issue embodied by A iff its informative content *coincides* with some possibility $\alpha \in A$. And once we assume this relationship between issue-resolving responses and possibilities, we are forced to conceive of propositions as persistent sets of possibilities. After all, suppose that $\alpha \in A$ and that R is a response whose informative content coincides with α. Then, given the assumed relationship between possibilities and issue-resolving responses, R must be an issue-resolving response. But then any response that provides even more information than R must also be an issue-resolving response, and this means, again given the assumed relationship between possibilities and issue-resolving responses, that any subset β of α must also be a possibility in A.

Given this more restricted notion of propositions as persistent sets of possibilities, the characterization of $\leq$ can in fact be further simplified. We said above that $A \leq B$ iff every possibility in A is contained in some possibility in B. But if propositions are persistent, we could just as well say that $A \leq B$ iff $A \subseteq B$. To see this, first suppose that $A \subseteq B$. Then clearly every possibility in A is contained in some possibility in B. Now, for the other direction, suppose that every possibility in A is contained in some possibility in B, and let α be a possibility in A. Then α must be contained in some possibility β in B. But then, since B is persistent, every subset of β, including α, must also be in B. So we must have that $A \subseteq B$.

Fact 2 (Further simplification of $\leq$) $A \leq B$ *iff* $A \subseteq B$

From this characterization it immediately follows that $\leq$ forms a partial order over Π. In particular, $\leq$ is anti-symmetric, which means that every two non-identical propositions really differ in informative and/or inquisitive content.

Fact 3 (Partial order) *$\leq$ forms a partial order over Π*

3.2 Algebraic operations

The next step is to see what kind of algebraic operations $\leq$ gives rise to. It turns out that, as in the classical setting, any two propositions A and B have a unique greatest lower bound (meet) and a unique least upper bound (join) w.r.t. $\leq$.

Theorem 1 (Meet).
For any two propositions A and B, $A \cap B$ is the meet of A and B w.r.t. $\leq$.

Proof. Clearly, $(A \cap B) \leq A$ and $(A \cap B) \leq B$, which means that $A \cap B$ is a lower bound of A and of B. What remains to be shown is that $A \cap B$ is the *greatest* lower bound of A and B. That is, for every C that is a lower bound of A and B, we must show that $C \leq (A \cap B)$. To see this let C be a lower bound of A and B, and let γ be a possibility in C. Then, since $C \leq A$, γ must be in A, and since $C \leq B$, γ must also be in B. But that means that γ must be in $A \cap B$. Thus, $C \leq (A \cap B)$, which is exactly what we set out to show. So $A \cap B$ is indeed the greatest lower bound of A and B. □

Theorem 2 (Join).
For any two propositions A and B, $A \cup B$ is the join of A and B w.r.t. $\leq$.

Proof. Clearly, $A \leq (A \cup B)$ and $B \leq (A \cup B)$, which means that $A \cup B$ is an upper bound of A and B. What remains to be shown is that $A \cup B$ is the *least* upper bound of A and B. That is, for every C that is an upper bound of A and B, we must show that $(A \cup B) \leq C$. To see this let C be an upper bound of A and B, and let α be a possibility in $(A \cup B)$. Then α must be in A or in B. Without loss of generality, suppose that $\alpha \in A$. Then, since $A \leq C$, α must also be in C. This establishes that $(A \cup B) \leq C$, which is what we set out to show. Thus, $A \cup B$ is indeed the least upper bound of A and B. □

As before, we will use $M(A, B)$ and $J(A, B)$ to denote the meet and the join of A and B, respectively. The existence of meets and joins implies that $\langle \Pi, \leq \rangle$ forms a lattice. And again, this lattice is bounded, i.e., there is a bottom element, $\{\emptyset\}$, and a top element, $\wp(W)$. For every proposition A, we have $\{\emptyset\} \leq A \leq \wp(W)$.

So far, then, everything works out just as in the classical setting. However, unlike in the classical setting, not every proposition has a complement, in the sense that not for every proposition A there is another proposition B such that (i) $M(A,B) = \{\emptyset\}$ and (ii) $J(A,B) = \wp(W)$. For instance, suppose that $W = \{w_1, w_2, w_3, w_4\}$ and let A be the proposition consisting of the possibilities α and β depicted in figure 1(a), and all subsets thereof. Then the only proposition B that satisfies (ii) is $\wp(W)$. But $\wp(W)$ does not satisfy (i). Thus, given this particular proposition A, it is impossible to find a proposition B that satisfies both (i) and (ii).

The fact that certain propositions do not have a complement in the above sense implies that $\langle \Pi, \leq \rangle$ does not form a Boolean algebra.

However, we will show below that for any two propositions A and B, there is a unique greatest proposition C such that $M(A,C) \leq B$. In algebraic jargon, this proposition is called the *pseudo-complement of A relative to B*, and the existence of relative pseudo-complements implies that $\langle \Pi, \leq \rangle$ forms a Heyting algebra (the algebra underlying intuitionistic logic).

Definition 5. *For any two propositions A and B, we define A/B as follows:*

$$A/B = \{\gamma \mid \text{for every } \chi \subseteq \gamma, \text{if } \chi \in A \text{ then } \chi \in B\}$$

Theorem 3 (Relative pseudo-complementation). *For any two propositions A and B, A/B is the pseudo-complement of A relative to B.*

Proof. First, let us show that for any A and B, $M(A, A/B) \leq B$. We know that $M(A, A/B)$ amounts to $A \cap A/B$. Now let ξ be a possibility in $A \cap A/B$. Then ξ is both in A and in A/B. Since $\xi \in A/B$, it must be the case that *if* $\xi \in A$ *then also* $\xi \in B$. But we know that $\xi \in A$. So ξ must also be in B. This establishes that $M(A, A/B) \leq B$.

It remains to be shown that A/B is the *greatest* proposition C such that $M(A,C) \leq B$. In other words, we must show that for any proposition C such that $M(A,C) \leq B$, it holds that $C \leq A/B$. To see this, let C be a proposition such that $M(A,C) \leq B$ and let γ be a possibility in C. Towards a contradiction, suppose that $\gamma \notin A/B$. Then there must be some $\chi \subseteq \gamma$ such that $\chi \in A$ and $\chi \notin B$. Since C is persistent, $\chi \in C$. But that means that χ is in $A \cap C$, while $\chi \notin B$. Thus $M(A,C) \not\leq B$, contrary to what we assumed. So A/B is indeed the pseudo-complement of A relative to B. □

In terms of relative pseudo-complementation, we can also define the pseudo-complement of a proposition A *simpliciter* (not relative to any other proposition B).

Definition 6 (Pseudo-complementation).
For every proposition A, we define the pseudo-complement of A, A^*, as the pseudo-complement of A relative to the bottom element of our algebra:

$$A^* = A/\{\emptyset\}$$

Fact 4 (Alternative characterization of A^*) $A^* = \{\overline{\bigcup A}\}$.

Thus, starting with a new notion of propositions and an order on these propositions which compares both the informative and the inquisitive content that they embody, we have established an algebraic structure with four operations on propositions, *meet*, *join*, and *(relative) pseudo-complementation*.

3.3 Connectives

Now suppose that we have a certain language L, whose sentences express the kind of propositions considered here. Then it is natural to assume that this language has certain sentential connectives which semantically behave like *meet*, *join* or *(relative) pseudo-complementation* operators. Below we define a semantics for the language of propositional logic, L_P, that has exactly these characteristics: conjunction behaves semantically like a *meet* operator, disjunction behaves like a *join* operator, negation behaves like a *pseudo-complementation* operator, and implication behaves like a *relative pseudo-complementation* operator. The semantics assumes a valuation function which assigns a truth-value to every atomic sentence in every world. For any atomic sentence p, the set of worlds where p is true is denoted by $|p|$.

Definition 7 (Algebraically motivated inquisitive semantics for L_P).

- $[p] = \{\alpha \mid \alpha \subseteq |p|\}$
- $[\neg\varphi] = [\varphi]^*$
- $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$
- $[\varphi \vee \psi] = [\varphi] \cup [\psi]$
- $[\varphi \to \psi] = [\varphi] \,/\, [\psi]$

Natural languages are of course much more intricate than the language of propositional logic. We expect, however, that natural languages will generally also have connectives which behave semantically as *meet*, *join*, and *(relative) pseudo-complementation* operators. For a linguistic case study based on this expectation, see [6].

3.4 Quantifiers

The approach taken here can straightforwardly be extended to obtain an inquisitive semantics for the language of first-order logic, L_{FO}. The universal quantifier can be taken to behave semantically as a generalized *meet* operator, which does not necessarily operate on just two propositions—like the *meet* operator considered above—but more generally on a (possibly infinite) set of propositions. Similarly, the existential quantifier can be taken to behave semantically as a generalized *join* operator.[1]

Fact 5 (Generalized meet) *For any non-empty set of propositions Σ, $\bigcap \Sigma$ is the greatest lower bound, i.e., the meet of all propositions in Σ w.r.t. $\leq$.*

[1] There are several alternative algebraic perspectives on quantification as well [7, 8]. In future work, we will explore these different perspectives in the inquisitive setting.

Fact 6 (Generalized join) *For any non-empty set of propositions* Σ, $\bigcup \Sigma$ *is the least upper bound, i.e., the join of all propositions in* Σ *w.r.t.* $\leq$.

As usual, the semantics for L_{FO} assumes a domain of individuals D and a world-dependent interpretation function I_w that maps every individual constant c to some individual in D and every n-place predicate symbol R to some n-tuple of individuals in D. Formulas are interpreted relative to an assignment function g, which maps every variable x to some individual in D. For every individual constant c, $[c]_{w,g} = I_w(c)$ and for every variable x, $[x]_{w,g} = g(x)$. An atomic sentence $Rt_1 \dots t_n$ is true in a world w relative to an assignment function g iff $\langle [t_1]_{w,g}, \dots, [t_n]_{w,g} \rangle \in I_w(R)$. Given an assignment function g, the set of all worlds w such that $Rt_1 \dots t_n$ is true in w relative to g is denoted by $|Rt_1 \dots t_n|_g$.

Definition 8 (Algebraically motivated inquisitive semantics for L_{FO}).

- $[Rt_1 \dots t_n]_g = \{\alpha \mid \alpha \subseteq |Rt_1 \dots t_n|_g\}$
- $[\neg\varphi]_g = [\varphi]_g^*$
- $[\varphi \wedge \psi]_g = [\varphi]_g \cap [\psi]_g$
- $[\varphi \vee \psi]_g = [\varphi]_g \cup [\psi]_g$
- $[\varphi \rightarrow \psi]_g = [\varphi]_g \,/\, [\psi]_g$
- $[\forall x.\varphi]_g = \bigcap_{d \in D} \; [\varphi]_{g[x/d]}$
- $[\exists x.\varphi]_g = \bigcup_{d \in D} \; [\varphi]_{g[x/d]}$

4 Comparison with previous work

In this section we briefly compare the algebraically motivated semantics presented here (i) with the inquisitive semantics for L_P specified in [1, 4, 5], (ii) with the inquisitive semantics for L_{FO} specified in [2], and (iii) with the 'unrestricted' inquisitive semantics for L_P and L_{FO} specified in [1, 3]. The comparison will be brief and not fully self-contained (familiarity with the cited work is necessary to understand and appreciate some of the claims made).

First, let us consider the inquisitive semantics for L_P specified in [1, 4, 5]. Even though this semantics was defined in different terms and motivated in a different way, it is entirely equivalent with the semantics presented here. Thus, as far as L_P is concerned, our algebraic considerations have not led to a new semantics, but rather to a more solid foundation for an existing system.

Next, let us consider the inquisitive semantics for L_{FO} specified in [2]. Interestingly, this semantics does *not* coincide with the one presented here, and the source of the differences between the two systems lies in the very basic notion of propositions that is assumed. Here we argued that, as long as we are interested in capturing informative and inquisitive content and nothing more than that, propositions should be defined as *persistent* sets of possibilities. In this way, we rule out the existence of propositions that are non-identical but equivalent in terms of informative and inquisitive content, such as those depicted in figure 1. Or in more technical terms, in this way we make sure that $\leq$ is *anti-symmetric*.

In [2], propositions are sets of possibilities such that no possibility is properly contained in a maximal possibility. That is, a proposition A cannot contain two possibilities α and μ such that μ is a maximal element of A and $\alpha \subset \mu$. This rules out, for instance, propositions like the one depicted in figure 1(b). It even assures that no proposition is equivalent but non-identical to the proposition depicted in figure 1(a). However, it does not generally rule out the existence of non-identical equivalent propositions. In other words, it does not make $\leq$ anti-symmetric.

In fact, several cases are discussed in [2] of sentences that are assigned different propositions, even though they are entirely equivalent in terms of informative and inquisitive content. This is presented as a desirable result, which indicates that the system in [2] is really intended to capture more than just informative and inquisitive content. However, it remains to be better understood what kind of content is supposed to be captured by the propositions in this system, besides informative and inquisitive content. Once this is better understood, it should be possible to define a natural order on those propositions, and to motivate the clauses of the semantics algebraically, as we have done here. For now, we conclude that the first-order system developed in the present paper is the most appropriate system as long as we are really only interested in informative and inquisitive content.[2]

Finally, let us consider the 'unrestricted' inquisitive semantics for L_P and L_{FO} specified in [1, 3]. These systems are called 'unrestricted' because they do not impose any restrictions on the notion of propositions that we started out with—that is, propositions are simply defined as arbitrary non-empty sets of possibilities—and they are very explicitly aimed at capturing more than just informative and inquisitive content. In particular, these systems are based on the idea that, besides informative and inquisitive content, propositions can also be taken to capture *attentive* content.

However, the clauses of these systems have so far not been motivated properly, and indeed, they have certain undesirable features (for instance, conjunction is not *idempotent*, i.e., it is not generally the case that $[\varphi \wedge \varphi] = [\varphi]$). Thus, it would be useful to extend the algebraic approach taken here also to the unrestricted setting. The crucial step in establishing such an extension would be to specify an attentiveness order, which determines when one proposition is more attentive than another. Given such an order, it may again be possible to define *meet*, *join*, and *complementation* operators, which can then be associated with

[2] A further connection worth mentioning is the following. The semantics for L_P specified in [1, 4, 5] is based on the notion of *support*, a relation between possibilities and formulas. The core of the semantics is a recursive definition of this support relation. Subsequently, the proposition expressed by a formula φ is defined as the set of all maximal possibilities supporting φ. In [2] it is argued that this approach can not be extended to the first-order setting. Thus, an alternative approach is developed, leading to a direct recursive definition of the proposition expressed by a formula, bypassing the notion of support. However, if we define the proposition expressed by a formula φ as the set of *all* possibilities supporting φ, rather than the set of *maximal* possibilities supporting φ, we obtain a suitable first-order system, in fact one that is equivalent to the system developed in the present paper.

connectives and quantifiers. This direction is currently being pursued, and initial results are reported in [9].

5 Conclusion

In this paper we have defined an inquisitive semantics for the language of propositional logic and first-order predicate logic, motivated by general algebraic concerns. We argued that propositions should be defined as persistent sets of possibilities, and we considered an order on such propositions which determines when one proposition is at least as informative and inquisitive as another. We showed that this order gives rise to a Heyting algebra, with *meet*, *join*, and *(relative) pseudo-complement* operators. Our semantics associates these semantic operators with connectives and quantifiers.

The semantics for L_P presented here is equivalent with the one specified in earlier work [1, 4, 5]. Thus, as far as L_P is concerned, our algebraic considerations did not lead to a new semantics, but rather to a more solid foundation for an existing system. In the case of L_{FO}, the semantics developed here did diverge from previous work [2] and we argued that, as long as we are only interested in capturing informative and inquisitive content, the present system is most appropriate. In future work, we hope to extend the approach also to the 'unrestricted' setting, where propositions do not only embody informative and inquisitive content, but also attentive content.

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